Two-Body System Conclusion and Reality Use: Slingshot Effect

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Abstract

In this thesis, a summary will be made for two-body problems, and the author tries to organize all these additional problems and two-body motion together to build a full system(from basic ideal situation to general situation, from Newton theory to Einstein Theory and from theoretical knowledge to reality use). After the construction of the system, slingshot effect will be introduced in the thesis. We will be able to discuss the possibility of space travel and the time cost. In addition, the main approach (or work), this paper will focus on theoretical analysis and math calculation, which shows that from mathematical views, all Kepler work could be achieved by serious calculation. (It means the combination of mathematic and astronomic observation.)

Keywords

Kepler Laws; Conservation of Angular Momentum; Conservation of Energy; Perihelion Near Movement; Gravitational Potential Energy; Slingshot Effect

Methods

In order to construct a full system, a clear logic is needed to be used (from special to general to organize this thesis). Page 2 is the main body of the thesis. The first part is theoretical analysis, the second part perihelion precession will be calculated and for the third part space, the travel way-Slingshot will be discussed.

In the first part: Kepler Second Laws will be proved, after this, a parameter equation is needed to deduce planet orbit. And discuss the shape of the orbits. This part will show the proven way of Kepler Third Laws and keep discuss another two-body orbit-binary system. Then a brief discussion will be made for a special binary system-two black hole merge and gravitational wave. As an expansion for the Two-Body System.

In the second part: A general situation will be considered, which other planet gravity will affect to orbit, and that is perihelion precession (calculate the correction with other planet potential energy). Einstein General and Special Relativity will be used to calculate the correction

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and compare the calculation with observation.

Then we will compare these 2 different corrections (Newton classical theory and Einstein theory) so that we can check different ways' contribution to perihelion precession.

In the third part: we will introduce the slingshot effect and the whole travel of Voyager 1(how it works, why it is good for space travel.) and we will make a discussion about the time we need to get to Uranus by using Jupiter as the accelerator. And combine with data from Voyager1, to identify the feasibility of the slingshot effect. By modeling and data of voyager 1, to discuss what's the short part of this study and why will have these errors.

Background: Historical Part

Johannes Kepler, who was born in 1571 at 12.27, passed in 1630 at 11.15. He is a splendid German astronomer, physicist and mathematician. Three important laws of planet motion had been discovered, which had an outstanding contribution to astrophysics. In fact, Kepler spent almost 20 years on data collection and analysis, which also provides evidence to the correction of gravitational law (Newton law).

In astrophysics, we can say that the most basic laws are Kepler laws. Obviously, people were not able to get data from the direct experiment, but need send detectors and telescope observation to achieve the goal. So lots of final results are based on inference. The two-body problem is the only problem that has a definite solution. Here are the three laws: The orbit of a planet is oval, and the sun's position is in one of the oval's focus. The area in which the planet radius passed in a unit time is a constant. The ratio of cubic of longer axis and planet period squared is a constant.

Later generations called him "Founder of celestial mechanics". The role of the 3 laws in astronomy is just as same as Newton's second law in physics.

Then in 1916, Einstein used general relativity to explained Mercury precession successfully, and this is one of the 3 astronomical identifications for his theory. His calculation result is $1°33'19.91''$ and the observed result is $1°33'20''$. It is pretty identical.

Voyager1 is designed by NASA. It was sent to space in 1977 on September 5th. It is the first spacecraft that is sent to Jupiter, Saturn and other planets. Voyager 1 also took lots of photos and details of those planets. Now, Voyager1 had gone out of the solar system.

Mathematical Part

The reason why the part will be listed is the mathematical knowledge we need is much more than the physics part.

Taylor expansion: A significant way to calculate complex function by approximately calculating and here we do not have to talk about the specific proof, the expansion can be shown below:

$$f(x) = C + \sum_{n=1}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

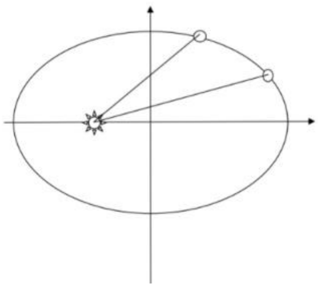
In the formula, C means a constant, in most situations, it is truly small. And "a" means the base point we will choose. "N" could not be infinity unless the function has infinity derivatives.

Lagrange formula: When we try to deal with a problem that has complex orbit or mechanism analysis, the Lagrange formula will be a useful tool.

Kepler second law's proof

Way 1

According to figure 1, the sun is in one focus of the oval, the angle of radius and the x-axis are changing with the radius length. For now, it needs to be assumed that angle and radius are both functions with time. So we got $\theta(t)$ and $r(t)$.



(Figure 1.1)

The motion can be seen as a circular motion so, in the whole process, there are no other forces except gravity, which means angular momentum is a constant. So angular momentum is conserved:

$$J = m\frac{d\theta}{dt}r^2 = C$$

At the same time, sector area is:

$$A = \frac{1}{2}\theta r^2 \ (1.1)$$

Then we got:

$$\frac{dA}{d\theta} = \frac{1}{2}r^2 (1.2)$$

From (1.1), angular speed equal to:

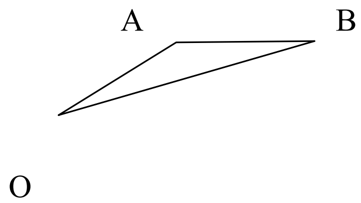
$$\frac{d\theta}{dt} = \frac{J}{mr^2} (1.3)$$

Then we need to multiply (1.2) and (1.3) together:

$$\frac{dA}{d\theta}\frac{d\theta}{dt} = \frac{dA}{dt} = \frac{J}{2m} = Const. (1.4)$$

Way 2

If Planet passed a distance Δr, in a short time, so the distance can be seen as a straight line. Shown out in this triangle:



(Figure 1.2)

In figure 2, the height of the triangle is $r(t_1)$, so the area is:

$$\Delta A = \frac{\overrightarrow{r(t)}}{2}\Delta r \ (1.5)$$

When the time is so short, then we got:

$$\frac{\Delta A}{\Delta t} = \frac{\Delta r}{\Delta t}\frac{\overrightarrow{r(t)}}{2} = \frac{\overrightarrow{\dot{r}(t)}}{2}\overrightarrow{r(t)} \ (1.6)$$

According to Newton Second Law and gravitational Law:

$$F = \frac{GMm}{\overrightarrow{r^2(t)}} \qquad \frac{d^2r}{dt^2} = \frac{GM}{\overrightarrow{r^3(t)}}\overrightarrow{r(t)} \ \ (1.7)$$

Calculate Second derivative:

$$\frac{d^2A}{dt^2} = \left\|\frac{1}{2}\overrightarrow{\dot{r}(t_i)\dot{r}(t_i)} + \overrightarrow{r(t_i)\ddot{r}(t_i)}\right\| \ (1.8)$$

$$\frac{d^2A}{dt^2} = \left(0 + \frac{GM}{\overrightarrow{r^3(t)}}\overrightarrow{r(t)r(t)}\right) = 0 \quad (1.9)$$

The second derivative equals 0 means the First derivative is a constant.

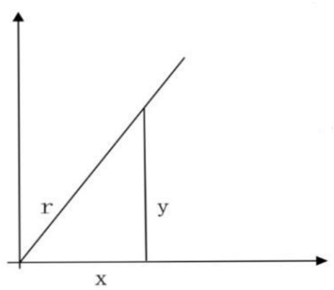
In addition, so far 2 different ways were used to prove Kepler's second Law, no matter use physics way or pure mathematics way, it could be easier to get the final result.

In conclusion, the essential key of Kepler's Second Law is the conservation of the angle momentum. Parameter equations were used to find the relationship between area and time.

Next, calculus is used to find the exact result, in the next content. We will use these thinking logics to solve problems. These are also the core ideas of this thesis.

Planet Motion Orbit: Star-Planet System

Because the motion of the planet is oval, polar coordinates need to be used to calculate.



(Figure 1.3)

This diagram shows velocity on r(t), and it components in x and y coordinates.

According to this diagram, we can get:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

It's simple to prove that:

$$\begin{cases} \frac{dx}{dt} = \dot{r}\cos\theta - \dot{\theta}\sin\theta \\ \frac{dy}{dt} = \dot{r}\sin\theta + \dot{\theta}\cos\theta \end{cases}$$

Add these 2 vectors together, get final velocity:

$$v^2 = (\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = \left(\dot{r}^2 + r^2\dot{\theta}^2\right)\ (2.0)$$

Because energy is conserved so we got:

$$E = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{GMm}{r}\ (2.1)$$

Because angular momentum is conserved so:

$$J = m\dot{\theta}r^2$$

So that 2 equations were shown out:

$$\begin{cases} \frac{d\theta}{dt} = \frac{J}{mr^2}\ (2.2) \\ \frac{dr}{dt} = \frac{\sqrt{2Emr + 2GMm^2r - J^2}}{mr}\ (2.3) \end{cases}$$

Use equation 1 to divide equation 2 then got:

$$\theta = \int \frac{J}{\sqrt{2GMm^2r^4 + 2Emr^3 - J^2r^2}} dr (2.4)$$

$$\begin{cases} a = 2GMm^2 \\ b = 2Em \\ c = J^2 \end{cases} \text{(just simply writting)}$$

$$\theta = \int \frac{J}{\sqrt{ar^4 + br^3 - cr^2}} dr\ (2.5)$$

$$\theta = \int \frac{J/r}{\sqrt{ar^2 + br - c}} dr$$

Let:

$$u = \frac{1}{r}$$

$$\theta = -\int \frac{\mathrm{J}u}{u^2\sqrt{\frac{\mathrm{a}}{\mathrm{u}^2}+\frac{\mathrm{b}}{\mathrm{u}}-\mathrm{c}}}\mathrm{du}$$

$$\theta = -\int \frac{\mathrm{J}/\sqrt{c}}{\sqrt{\frac{\mathrm{a}}{\mathrm{c}}+\frac{\mathrm{b}}{\mathrm{c}}\mathrm{u}-\mathrm{u}^2}}\mathrm{du}$$

Rewritten it:

$$\theta = -\int \frac{\mathrm{J}/\sqrt{c}}{\sqrt{\mathrm{m}^2-(u+n)^2}}\mathrm{du}$$

Calculate it then we got:

$$\theta = \mathrm{J}/\sqrt{c} \times \arccos\left(\frac{u+n}{m}\right) + C$$

$$\mathrm{m}^2 - (u+n)^2 = -u^2 - 2nu + (m^2 - n^2)$$

So:

$$\begin{cases} -2n = \dfrac{b}{c} \\ m^2 - n^2 = \dfrac{a}{c} \end{cases}$$

So that m and n values can be shown:

$$\begin{cases} n = -\dfrac{b}{2c} \\ m = \dfrac{\sqrt{b^2+4ac}}{2c} \end{cases}$$

$$\theta = \arccos\left(\frac{u}{m}+\frac{n}{m}\right) + C$$

We take the polar axis as the origin of calculation, which means we can make the constant value be 0:

$$\cos\theta = \frac{2c}{\sqrt{b^2+4ac}}u - \frac{b}{\sqrt{b^2+4ac}}$$

Because of:

$$\begin{cases} \mathrm{a} = 2\mathrm{GMm}^2 \\ \mathrm{b} = 2\mathrm{Em} \\ \mathrm{c} = \mathrm{J}^2 \end{cases}$$

Calculate and simplified so the final result can be got:

$$\frac{\mathrm{L}}{\mathrm{r}} = 1 + \mathrm{e}\cos\theta \quad (2.6)$$

(Here we make perihelion be the polar axis so that integration constant can be 0.)

$$\mathrm{e} = \sqrt{\frac{2\mathrm{EJ}^2}{\mathrm{m}^3\mathrm{M}^2\mathrm{G}^2}+1} \quad (2.7)$$

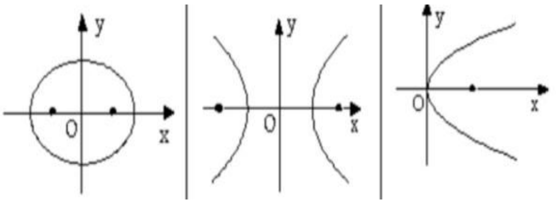
$$\mathrm{L} = \frac{\mathrm{J}^2}{\mathrm{m}^2\mathrm{GM}} \quad (2.8)$$

"e" here means eccentricity, according to dimension analysis, the unit of this is 1, it's also proved that.

We need to pay attention to the term of energy, which is “E”. The value of eccentricity depends on the energy. Here we got 4 kinds of orbits:

1. When the eccentricity value is equal to 0, the orbit is a circle.
2. When the eccentricity value is between 0 and 1, the orbit is an oval.
3. When the eccentricity value is equal to 1, the orbit will be a parabola.
4. When the eccentricity value is bigger than 1, the orbit will be a hyperbola curve.

But first orbit shape seems almost impossible in the universe, because it is hard for energy to get a so exact value, and planet velocity needs to be a constant. So general orbits are shown in figure 1.4:



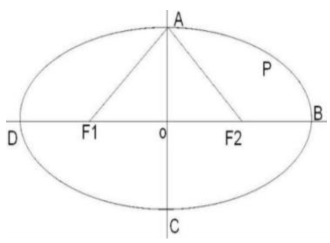
(Figure 1.4)

These orbit shapes all depend on eccentricity, we need to pay attention to the hyperbola curve, its shape of orbit can be used to make spacecraft get a higher velocity, which is the "slingshot effect".

Kepler Third Law Proof

Now we have got the planet motion formula, it is the conic curve, and most two-body orbits are oval, orbit shape has already shown Kepler first law: The orbit of a planet is oval, and the sun's position is in one of the oval's focus (the way here is from Theoretical Mechanics page 53).

In addition, we need to get some basic information: perihelion, aphelion, semi-major axis and semi-minor axis.



(Figure 1.5)

In figure 1.5, F1 and F2 are 2focuses of this oval, DF1 length is perihelion length, and F1B length is aphelion length, AF1+AF2=DF1+F1B.(it comes from the website.)

When $\theta = 0$, we can get perihelion length:

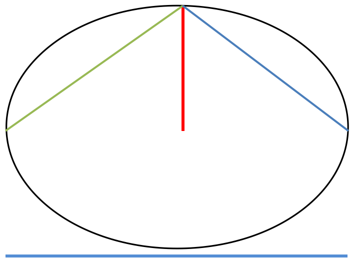
$$r_{min} = \frac{L}{1+e} = OB - OF_2 = OB(1-e) \ (2.9)$$

And aphelion length:

$$r_{max} = \frac{L}{1-e} = OB(1+e) \ (3.0)$$

And semi-major axis length:

$$OB = OD = \frac{L}{1-e^2} \ (3.1)$$



(Figure 1.6)

According to the definition of the oval, we can say the sum of the two green lines is equal to the sum of perihelion length and aphelion length. Let the red line be "b". The green line is "c" half of the blue line be "a". Then we got:

$$2c = \frac{L}{1-e} + \frac{L}{1+e} = \frac{2L}{1-e^2}$$

$$c = \frac{L}{1-e^2}$$

$$a = OBe = \frac{Le}{1-e^2}$$

$$b^2 = c^2 - a^2 = \frac{L^2(1-e^2)}{(1-e^2)^2}$$

$$b = \frac{L}{\sqrt{1 - e^2}}$$

The period of the planet can also be seen as the time which the radius passed through the whole oval. So we got:

$$T = \frac{\pi ab}{dA/dt} = \frac{2\pi ab}{r^2\dot{\theta}} = \frac{\frac{2\pi L^2}{(1-e^2)^{\frac{3}{2}}}}{J/m} \quad (3.2)$$

Simplified so we got:

$$T = 2\pi a^{\frac{3}{2}}\sqrt{\frac{1}{GM}} \quad (3.3)$$

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM} = \text{Const} \quad (3.4)$$

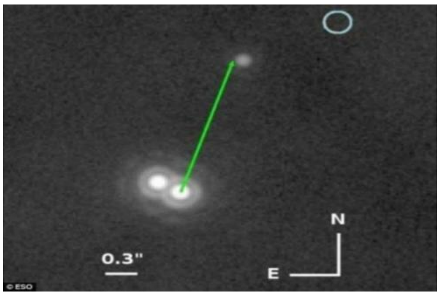
Kepler's Third Law has been proved successfully.

In addition, we could use (1.8) and (1.9) to show the semi-major axis and semi-minor axis, they can be shown like:

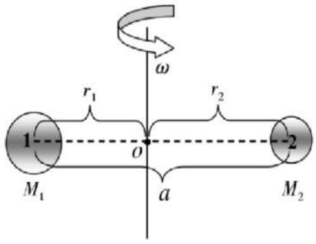
$$\begin{cases} a = \dfrac{GMm}{2|E|} & (3.5) \\ b = \dfrac{J}{\sqrt{2m|E|}} & (3.6) \end{cases}$$

Binary System: Basic Information

This situation is more general compared with star-planet orbit, in star-planet orbit $M \gg m$, so M can be nearly seen as the center of orbit, but in the binary system, the situation will be: $M > m$ or M=m, so the problem will get harder.



This figure shows a planet rotated with 2 stars. This photo was taken from a very far galaxy. It can help astronomer study how the binary system formed. (Photo was taken by European Southern Observatory in Chile 2013 1st April.)



This figure shows 2 stars rotated each other, O is the center of rotation, r1 is M1 rotational radius, r2 is M2 rotational radius (the diagram is from An Analysis of Binary Star System in high School Physics by Lianghan Lin in 2020).

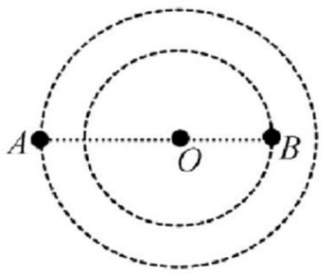
We take O as the center of the coordinates, two stars line is the x-axis, in order to simplify, let:

$$\begin{cases} x_1 = (r_1, 0, 0) \\ x_2 = (-r_2, 0, 0) \end{cases}$$

So that we have list relationship:

$$\begin{cases} r_1 = \dfrac{M2}{M1 + M2} a & (3.7) \\ r_2 = \dfrac{M1}{M1 + M2} a & (3.8) \end{cases}$$

Here we can assume that the orbit of 2 stars is a circle. Orbits are shown below diagram:



This diagram shows two stars orbits, B is the smaller mass orbit, A is the larger one (the diagram is from An Analysis of Binary Star System in high School Physics by Liang han Lin in 2020)

The orbit shape is 2 circles, in addition, we can get other information except this. Gravity is approximately equal to centripetal force: For M1:

$$\frac{GM1M2}{r_1^2} = \frac{M1}{r_1}{v_1}^2 \ (3.9)$$

For M2:

$$\frac{GM1M2}{r_2^2} = \frac{M2}{r_2}{v_2}^2 \ (4.0)$$

We can get their line speed:

$$\begin{cases} v_1 = \sqrt{\frac{GM2}{r_1}} \\ v_2 = \sqrt{\frac{GM1}{r_2}} \end{cases}$$

Simplified then we got:

$$v_1 = v_2 = \sqrt{\frac{G(M1+M2)}{a}} \ (4.1)$$

But for angular speed, we have $\omega = \frac{v}{r}$:

$$\begin{cases} \omega_1 = \sqrt{\frac{G(M1+M2)}{ar_1^2}} = \sqrt{\frac{G(M1+M2)^3}{M2^2a^3}} \ (4.2) \\ \omega_2 = \sqrt{\frac{G(M1+M2)}{ar_2^2}} = \sqrt{\frac{G(M1+M2)^3}{M1^2a^3}} \ (4.3) \end{cases}$$

There is a special situation, if angular speeds are the same, which means two stars are relatively rest, then angular speed will be:

$$\omega^2 = \frac{G(M1+M2)}{a^3} \ (4.4)$$

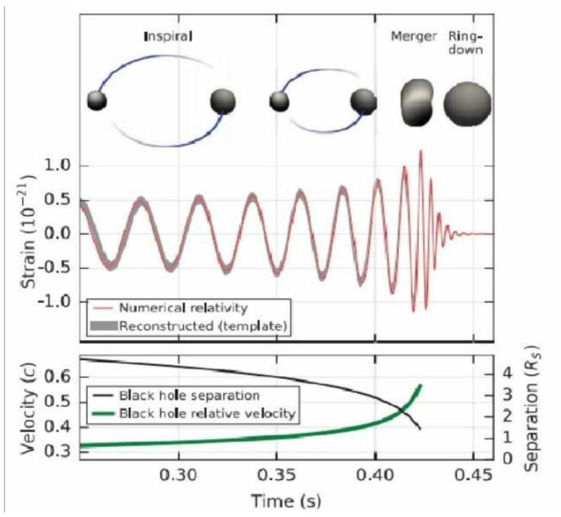
Extra Part-Black Hole Merge

In this part, we will make a simple discussion for 2 massive black holes merge, which will cause a gravitational wave.

According to Einstein's theory, two massive objects (stars) will cause a strain for both time and space. Shown by the formula below:

$$t_a = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}t$$

Black holes with higher speed, the slower time.



This diagram shows the data and analysis results from the LIGO team. We can see that, when 2 black holes get closer to each other, the strain for space is really small, only between $\pm 1 \times 10^{-21}$ units, and the speed is pretty fast, which is 0.6 times of light speed! (the diagram is from the Evolution of Blackhole and Relevant Issues of Gravitational Waves on May 27, 2017, by Zhong wen Feng)

So we could calculate the maximum time value, when t=1 year. Relative time will be:

$$t_a = \frac{1}{\sqrt{1-0.36}} \times 1\text{year} = 1.5625\text{years}$$

In conclusion, black hole merge is an extreme situation, and gravitational waves only can be detected when 2 massive black holes merge.

Perihelion Near Movement: Potential Correction

In the solar system, the planet's orbit isn't a closed oval because of the distraction of other planets' gravities. This effect can be seen as another potential energy forced on the planet. It could be assumed that this potential energy is $\delta V(r)$. So:

$$V_{(r)} = -\frac{GMm}{r} + \delta V_{(r)} \quad (4.5)$$

When this planet moves a period, the angle of near movement is shown in this function:

$$\Delta\theta = 2\int_{r_{min}}^{r_{max}} \frac{Jdr}{\sqrt{ar^4 + br^3 - cr^2 - 2mr^4\delta V_{(r)}}}$$

The angle of near movement is caused by $\delta V_{(r)}$, so we can consider $\Delta\theta$ is a function of $\delta V_{(r)}$, now use Taylor expansion to make approximate calculation (expansion is calculated in $\delta V(r)=0$ and we only calculate to the first derivative):

$$\Delta\theta(\delta V_{(r)}) \approx \Delta\theta(0) + \Delta\dot{\theta}(0)\delta V(r) = 2\pi + \delta\theta$$

In addition, 2π is the contribution of this potential energy:

$$V(r) = -\frac{GMm}{r}$$

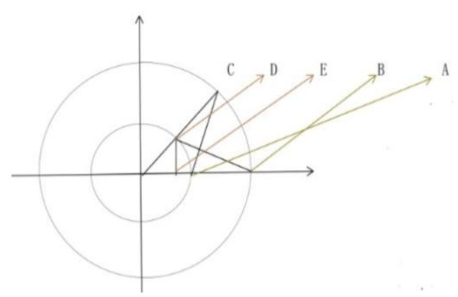
The formula of near movement is shown like this (formula 4.6 and 4.7 and calculation result from Theoretical Mechanics page 57):

$$\delta\theta = \frac{\partial}{\partial J}\int_{r_{min}}^{r_{max}} \frac{2mdr}{\sqrt{ar^4 + br^3 - cr^2}}\delta V(r) \quad (4.6)$$

And it can also be shown by integration of angle:

$$\delta\theta = \frac{\partial}{\partial J}\left(\frac{2m}{J}\int_0^{\pi} r^2\delta V(r)d\theta\right) \quad (4.7)$$

Now, we need to calculate the value of the potential energy, one of the good examples is the Mercury movement. Here is the position of the planet (on point A, small circle is his orbit.)



(Figure 1.7)

In figure 1.7, the furthest planet orbit is the big circle and other planet orbits all between the small and big circle. We can assume that these planets' gravitational potential energy influences are distributed on the annulus. The big circle radius is a. Small circle radius is r. (eccentricity can be nearly seen as 0.) In order to find the potential energy, we need to find the distance, which is the length of AC:

In ΔOCA and ΔOBD:

$$\begin{cases} OA = OD \\ \angle COB = \angle COB \\ OC = OB \end{cases}$$

$$\Delta OCA \cong \Delta OBD$$

So AC can be shown as:

$$AC = DB = \sqrt{(a-x)^2 + y^2}$$

$$= \sqrt{a^2 + r^2 - 2ar\cos\theta}$$

So potential energy can be shown:

$$V(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} -\frac{GM}{\sqrt{a^2 + r^2 - 2ar\cos\theta}} d\theta \ (4.8)$$

Calculate it then we get:

$$V(r) \approx -\frac{GM}{a} - \frac{GMm}{4a^3} r^2 \ (4.9)$$

In this equation, the first term is normal potential energy, and the second term is $\delta V(r)$ we need to find, then we can calculate:

$$\delta\theta = -\frac{\partial}{\partial J}\left(\frac{2m}{J}\int_{0}^{\pi} \frac{GMm}{4a^3} r^4 d\theta\right) \ (5.0)$$

Then put formula (2.6) in this equation, so we can get the final result:

$$\delta\theta = \frac{3\pi}{2}\sqrt{1 - e_x^2}\left(\frac{M_y}{M}\right)\left(\frac{a_x}{r_y}\right)^3 \ (5.1)$$

e_x is the eccentricity of the object planet. M_y is the mass of the planet we need to consider. r_y is the radius of the planet we need to consider. (They are independent variable). M is the mass of the sun. a_x is the semi-major axis of the planet. (They are dependent variable).
Now, we have got the angle of near movement successfully.

Planetary Fact Sheet - Metric

	MERCURY	VENUS	EARTH	MOON	MARS	JUPITER	SATURN	URANUS	NEPTUNE	PLUTO
Mass (10^{24}kg)	0.330	4.87	5.97	0.073	0.642	1898	568	86.8	102	0.0146
Diameter (km)	4879	12,104	12,756	3475	6792	142,984	120,536	51,118	49,528	2370
Density (kg/m^3)	5427	5243	5514	3340	3933	1326	687	1271	1638	2095
Gravity (m/s^2)	3.7	8.9	9.8	1.6	3.7	23.1	9.0	8.7	11.0	0.7
Escape Velocity (km/s)	4.3	10.4	11.2	2.4	5.0	59.5	35.5	21.3	23.5	1.3
Rotation Period (hours)	1407.6	-5832.5	23.9	655.7	24.6	9.9	10.7	-17.2	16.1	-153.3
Length of Day (hours)	4222.6	2802.0	24.0	708.7	24.7	9.9	10.7	17.2	16.1	153.3
Distance from Sun (10^6 km)	57.9	108.2	149.6	0.384*	227.9	778.6	1433.5	2872.5	4495.1	5906.4
Perihelion (10^6 km)	46.0	107.5	147.1	0.363*	206.6	740.5	1352.6	2741.3	4444.5	4436.8
Aphelion (10^6 km)	69.8	108.9	152.1	0.406*	249.2	816.6	1514.5	3003.6	4545.7	7375.9
Orbital Period (days)	88.0	224.7	365.2	27.3*	687.0	4331	10,747	30,589	59,800	90,560
Orbital Velocity (km/s)	47.4	35.0	29.8	1.0*	24.1	13.1	9.7	6.8	5.4	4.7
Orbital Inclination (degrees)	7.0	3.4	0.0	5.1	1.9	1.3	2.5	0.8	1.8	17.2
Orbital Eccentricity	0.205	0.007	0.017	0.055	0.094	0.049	0.057	0.046	0.011	0.244
Obliquity to Orbit (degrees)	0.034	177.4	23.4	6.7	25.2	3.1	26.7	97.8	28.3	122.5
Mean Temperature (C)	167	464	15	-20	-65	-110	-140	-195	-200	-225
Surface Pressure (bars)	0	92	1	0	0.01	Unknown*	Unknown*	Unknown*	Unknown*	0.00001
Number of Moons	0	0	1	0	2	79	82	27	14	5
Ring System?	No	No	No	No	No	Yes	Yes	Yes	Yes	No
Global Magnetic Field?	Yes	No	Yes	No	No	Yes	Yes	Yes	Yes	Unknown
	MERCURY	VENUS	EARTH	MOON	MARS	JUPITER	SATURN	URANUS	NEPTUNE	PLUTO

(Figure 1.8)

Figure 1.8 shows data from different planets in the solar system (it is from the goggle website.)

We could calculate Venus influence for Mercury:

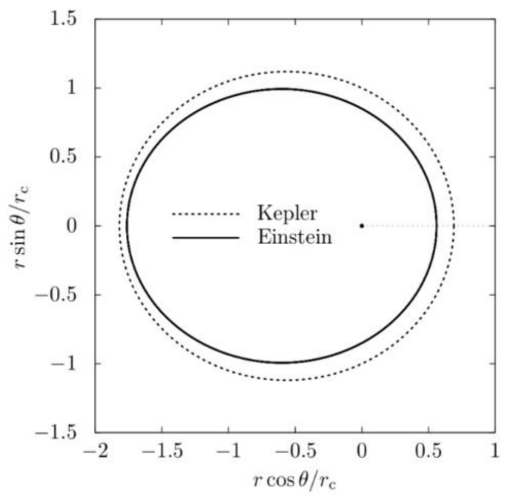
$$\delta\theta = 4.606\left(\frac{4.9 \times 10^{24}}{2 \times 10^{30}}\right)\left(\frac{5.8 \times 10^7}{10.8 \times 10^7}\right)^3$$

$\delta\theta = 151.22$ (arc second/century). In the same way, we also can calculate the other 7 planets' gravitational energies to the Mercury orbit effect, add them together, we can get the final result; 530 seconds of arc/century.

In conclusion, until now, we have used potential energy influence to get the effect to the near movement of Mercury. This is based on classical newton theory. But all these ways depend on the mass of planets that were distributed averagely on the ring. This way generally does not have a big influence, so the result is just approximately right.

Special Relativity Correction

In this part, we are going to use special relativity to correct the angle of near movement.



(Figure 1.9)

Figure 1.9 shows two theories, 2 orbit shapes.

In perihelion and aphelion, Einstein orbit has slightly different from Kepler orbit, we will find perihelion angle movement value in special relativity. To be specific, this figure is an x-y coordinate system observation the space movement is in z-dimension. (It from Kepler's Orbits and Special Relativity in Introductory Classical Mechanics, April 21, 2016, by Tyler J. Lemmon* and Antonio R. Mondragon.)

In fact, this is also the most classical identification for Einstein Theory. What we will do, is to find the orbit of the planet by relativity, and compare it with classical Kepler orbit (shown in formula 1.7).

In relativity, the most important factor is light speed, which is c. It is a limitation to speed, so in special relativity, kinetic energy will be shown:

$$E_k =- mc^2\sqrt{1-\frac{v^2}{c^2}} \quad (5.2)$$

Momentum will be shown like:

$$P = \frac{\partial E_k}{\partial v} = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} \quad (5.3)$$

Lagrange value is shown below:

$$L =- mc^2\sqrt{1-\frac{v^2}{c^2}} + \frac{GMm}{r} \quad (5.4)$$

To simplify the calculation, we use light speed be the unit, so $- mc^2 =- m$.

In order to find the correction, we need to expand it and consider 1 to 3 derivatives. Use Taylor expansion (do not conclude constant value):

$$L \approx \sum_{n=0}^{3} \frac{L^n(0)}{n!}(v-0)^n + \frac{GMm}{r} \quad (5.5)$$

$$L = \frac{mv^2}{2}\left(1+\frac{v^2}{4}\right) + \frac{GMm}{r} \quad (5.6)$$

At the same time, polar coordinates will be used to show velocity out (because angle and radius relationship is the function needed to be found):

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

(The energy does not conclude rest energy.)

So there will be angular momentum and momentum values, they are:

$$\begin{cases} J = \dfrac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}\left(1 + \dfrac{\dot{r}^2 + r^2\dot{\theta}^2}{2}\right) (5.7) \\ P = \dfrac{\partial L}{\partial \dot{r}} = m\dot{r}\left(1 + \dfrac{\dot{r}^2 + r^2\dot{\theta}^2}{2}\right) \ (5.8) \end{cases}$$

Energy can be shown below:

$$E = \frac{\partial L}{\partial \dot{\theta}} \times \frac{dr}{dt} + \frac{d\theta}{dt}\frac{\partial L}{\partial \dot{\theta}} - L \ (5.9)$$

$$E = m(\dot{r}^2 + r^2\dot{\theta}^2)\left(1 + \frac{v^2}{2}\right) - \frac{GMm}{r} \ (6.0)$$

According to the conservation of momentum in relativity, we have angular speed:

$$\frac{d\theta}{dt} = \frac{J}{mr^2(1 + \frac{v^2}{2})} \ (6.1)$$

The way is the same as finding Kepler orbit in classical theory:

$$\frac{d\theta}{dr} \times \frac{d\theta}{dt} = \frac{dr}{dt}$$

Make substitution, $\frac{dr}{dt}$ can be shown:

$$\dot{r} = \frac{J}{mr^2\left(1 + \frac{v^2}{2}\right)}\left(\frac{d\theta}{dr}\right) \ (6.2)$$

Until now, there is a set of equations have:

$$\begin{cases} \dfrac{dr}{dt} = \dfrac{J}{mr^2\left(1 + \frac{v^2}{2}\right)}\left(\dfrac{d\theta}{dr}\right) \ (6.3) \\ \dfrac{d\theta}{dt} = \dfrac{J}{mr^2(1 + \frac{v^2}{2})} \qquad (6.4) \end{cases}$$

So kinetic energy will be shown:

$$E_{k.} = \frac{J}{3rv}\left[\frac{2}{3r^2v^2}\left(\frac{d\theta}{dr}\right)^2 + 1)\right] \ (6.5)$$

$$E = E_k + V(r)$$

The total energy can be shown in:

$$E = \frac{J^2}{2m}\left[\left(\frac{d\theta}{dr}\right)^2 + \frac{1}{r^2}\right] - \frac{mv^4}{8} - \frac{GMm}{r} \ (6.6)$$

The correction of relativity is the second term.

$$E_{k.} = E + V(r) = \frac{mv^2}{2} \ (6.7)$$

$$v^2 = \frac{2E + 2V(r)}{m} \ (6.8)$$

$$\text{relativity corecction} = -\frac{mv^4}{8}$$

substituted and simplified the result:

$$-\frac{mv^4}{8} = -\frac{[E + V(r)]^2}{2m}$$

So we have equation below:

$$E(1 + \rho) + \frac{GMm}{r}(1 + 2\rho) = \alpha\left[\left(\frac{d\theta}{dr}\right)^2 + \frac{\beta}{r^2}\right]$$

$$\begin{cases} \rho = \dfrac{E}{2mc^2} \\ \alpha = \dfrac{J^2}{2m} \\ \beta = 1 - \dfrac{1}{(JcGMm)^2} \end{cases}$$

Simplified the equation, compared with Kepler orbit:

$$\begin{cases} \dfrac{d\theta}{dr} = \sqrt{-\beta r^{-2} + A(1+2\rho)r^{-1} + C} \; (6.9) \\ \dfrac{d\theta}{dr} = \dfrac{J}{\sqrt{2GMm^2r^4 + 2Emr^3 - J^2r^2}} \quad (7.0) \end{cases}$$

According to complex calculate, the final result is shown by the formula below:

$$\delta\theta = \frac{\pi GM}{a(1-e^2)c^2} \; (7.1)$$

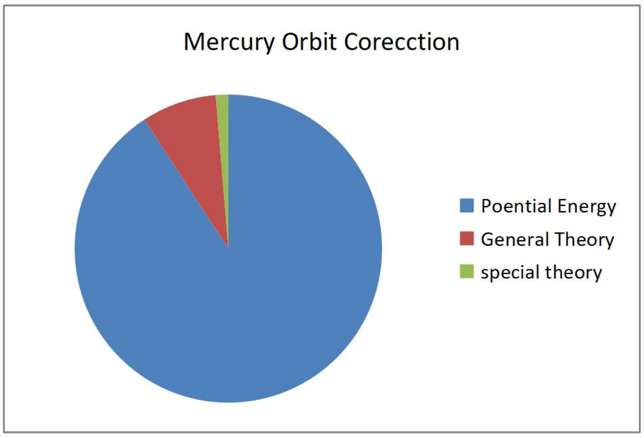
Calculating Mercury angular near movement so we get the angle is 7.17 seconds of arc per century. General relativity correction is 43 seconds of arc per century. It contains special relativity correction.

(In addition, this part of logical ways and final result are referenced from university text books-Theoretical Mechanics page 79, because there are too many complex calculations and it truly needs sophisticated knowledge.)

In addition, the general relativity correction is (the result is from Mercury's Perihelion by Chris Pollock on March 31, 2003):

$$\delta\theta = \frac{24\pi^3a^2}{cT^2(1-e^2)}$$

If we use this to calculate Mercury perihelion, the angle value will be 43 arc seconds per century. And observation value is 570 units, our calculation is 537.5 units. The reason why the error range is big is that our potential energy correction is based on the effect is averagely distributed, but as an approximate calculation, this value is enough.



(Figure 2.0)

Figure 1.9 shows 3 different theories for Mercury orbit correction, it is found that potential energy contribution is huge, it takes almost 92% part, the rest 8% is all from relativity, in addition, the reason why we say Mercury near movement is the most classical identification for relativity is that when we add all the gravitational potential effects together, the value only takes 92% percentage of the observation value, but when we use relativity to calculate, the result is almost same as that 8% result. And special relativity correction is contained by general relativity correction.

In conclusion, in this part we use Special Relativity to correct the Mercury near movement, from this result we can find, the effect of relativity is really small, most of the effects from the other planet gravitational potential energy.

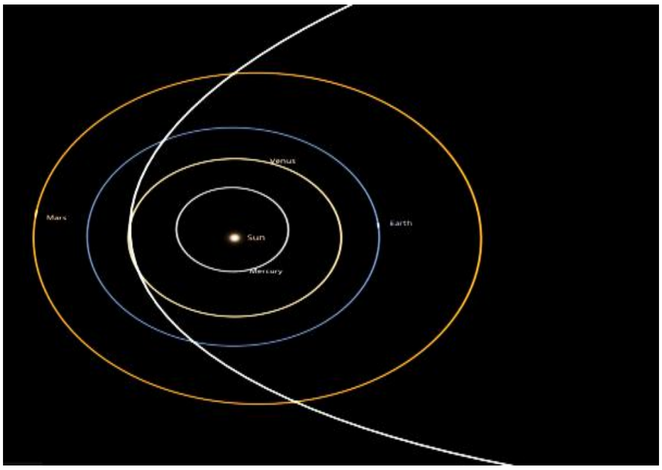
Slingshot Effect

The slingshot effect is a way to accelerate the detector by using massive planet gravity.

In 1918 Yuri, a soviet scientist first mentioned the slingshot effect in his paper (For those interested in building interstellar rockets).In 1959, it was first used by "Moon 3" (Russian detector) and take photos there.

Here we will introduce Voyager1 because it has the most classical slingshot. In part 3.4, we will use the data of Voyager1 to identify our calculation. The whole travel was divided into 5 steps:

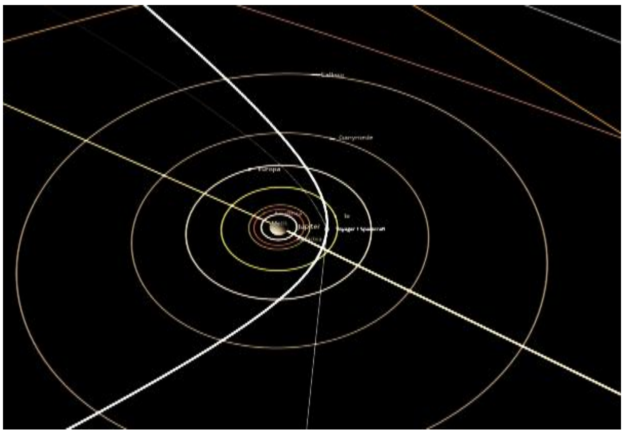
Step 1:



(Figure 2.1)

Voyager1 was sent to Venus orbit from earth, then travel to Jupiter.

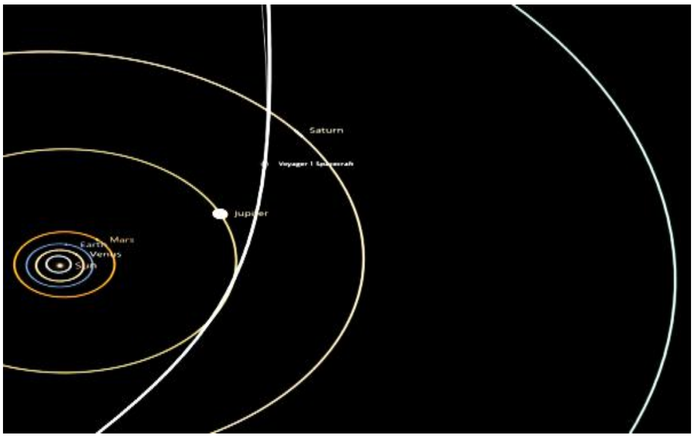
Step 2:



(Figure 2.2)

When the detector got Jupiter orbit, it use Jupiter as an accelerator to accelerate to reduce the consumption of the fuel and changing the direction of the detector. Energy is positive and it is a hyperbola curve.

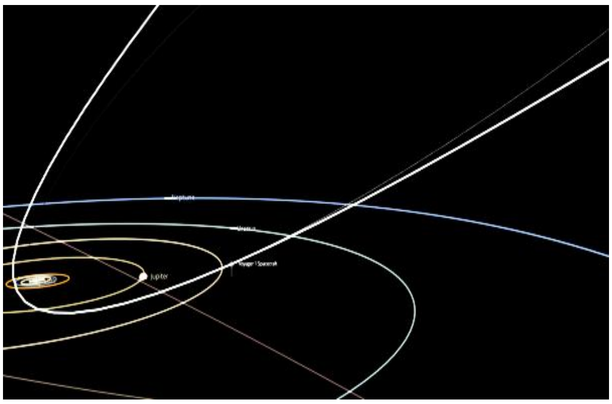
Step 3:



(Figure 2.3)

The detector uses Jupiter gravity to correct orbit in the order sent it to Saturn.

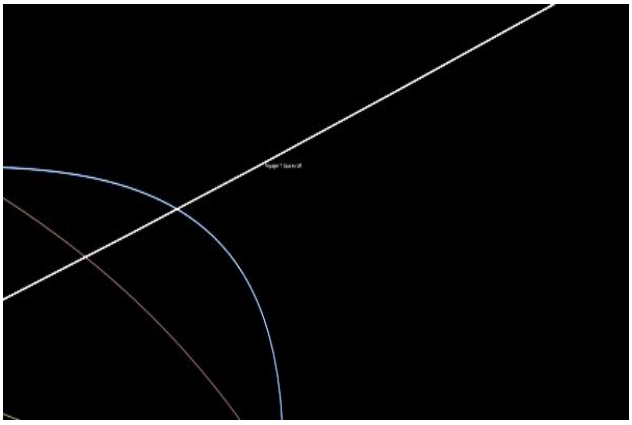
Step 4:



(Figure 2.4)

Voyager1 uses Saturn gravity to change the direction of its orbit to make orbital transfer vertically.

Step 5:



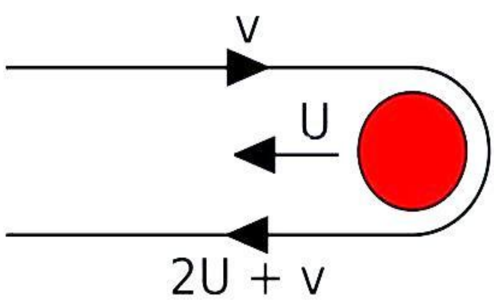
(Figure 2.5)

The detector will travel out of the solar system in this direction (But according to new data of NASA, they are still not sure if voyager 1 fly out of the solar system, in other words, the solar system may be much bigger than our thought.)

Principle

The planet rotates around the sun, so the planet got a speed. When the detector goes into the gravitational field of this planet, the detector will move with the planet, it will get a part of planet speed, which means the speed of the detector will go up. In the whole process, angular momentum and energy are conserved.

Just like this simple diagram shows:



(Figure 2.6)

V is the speed of the detector, U is the speed of the planet, when the detector move with the planet, the relative speed to the planet is still V, but the relative speed to the sun will be 2U+V. (this model is extremely easy, just for principle show.) (The graph is from the website).

Study Assumption

Here we assume that the detector from the earth needs to go to Uranus, and it needs to make the consumption of fuel minimum. And during our calculation, we will still assume these planets which we needed were all in the suitable orbit.

This figure shows planet distribution in the solar system. (It is from the website)

From the graph, we can find that detector will pass through Jupiter in the path from earth to Uranus.

In the 3.3.1 part, we will assume that detector did not pass Jupiter and find the time it needs to get to its destination, and calculate the minimum fuel it needs.

In the 3.3.2 part, we will assume that detector will use Jupiter as an accelerator. And find the time need and minimum fuel it needs to get to Uranus (use slingshot effect).

Study Purpose

1. According to compare 2 periods of time, find the time and fuel we can save if we use the slingshot effect.

2. Make a brief discussion of the limitation of the effect.

Travel With No Jupiter

In the whole process, it is important to distinguish the reference coordinate system.

Gravity is the centripetal force when the earth is moving around the sun:

$$\frac{mv_E^2}{r_E} = \frac{GMm}{r_E^2}$$

$$v_E = \sqrt{\frac{GM}{r_E}} \approx 29.8\text{km/s}$$

If we want to send the detector off the earth, kinetic energy will all change to potential energy then:

$$E_k = E_g = \frac{mv_D^2}{2} = \frac{GMm}{r_E}$$

Then detector speed will be shown:

$$v_D = \sqrt{2}v_E = 42.14\text{km/s} \ (7.2)$$

This value is we take the sun as coordinate system origin point if we take the earth as the origin point, speed will be:

$$v_D = 12.34\text{km/s}$$

We have two-body energy formula:

$$E(r,\theta,t) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{GMm}{r} \ (7.3)$$

Shown angle in the form of radius to reduce variable:

$$E(r,t) = \frac{1}{2}m\left(\dot{r}^2 + \frac{J^2}{m^2r^2}\right) - \frac{GMm}{r} \ (7.4)$$

To make E minimum, so we got:

$$\frac{\partial E}{\partial t} = 0$$

So we get:

$$\dot{r}\left(m\ddot{r} - \frac{J^2}{mr^3} + \frac{GMm}{r^2}\right) = 0 \ (7.5)$$

The linear acceleration is centripetal acceleration:

$$\ddot{r} = \frac{v^2}{r}$$

Simplified then we get:

$$\dot{r}\left(\frac{2E + 2GMm}{r^2} - \frac{J^2}{mr^3} + \frac{GMm}{r^2}\right) = 0$$

Linear speed can't be 0, so we got:

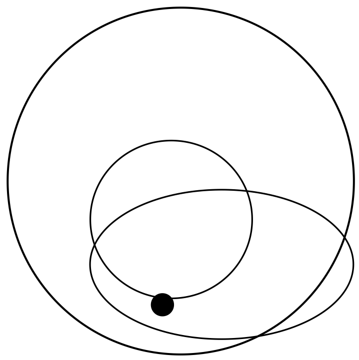
$$\frac{2Emr + 3GMm^2r - J^2}{mr^3} = 0$$

$$E = \frac{J^2}{2mr} - \frac{3GMm}{2} \ (7.6)$$

Until here, the derivative for radius makes the negative value of potential energy be the positive value of gravity. Whereas the detector's speed direction is opposite to the gravity, so displacement will be negative. Which means energy will be negative. And eccentricity formula is:

$$e = \sqrt{\frac{2EJ^2}{m^3M^2G^2} + 1}$$

If energy is negative, the e value will between 0 and 1, (or in other words, the speed of the detector did not reach the speed of solar system escape velocity so the detector will not get out from the solar system.) Which means the orbit shape is an oval, shown in the diagram:



(Figure 2.8)

The small circle is Earth orbit and the big circle is Uranus orbit, oval is the orbit of the detector. The black dot is the sun, and the sun is the focus of the oval.

The eccentricity of the 2 planet orbit can be seen approximately as 2 circles.

The radius of the earth is 1AU, the radius of Uranus is 19.2AU.

It is easy to find that radius of the earth is perihelion and Uranus radius is aphelion. So we got:

$$\begin{cases} r_{min} = \frac{L}{1+e} = 1AU \\ r_{max} = \frac{L}{1-e} = 19.2AU \end{cases}$$

Solve these equations:

$$\begin{cases} e = 0.9 \\ L = 1.9AU \end{cases}$$

We could get the equation of the detector:

$$r = \frac{1.9AU}{1 + 0.9\cos\theta} \quad (7.7)$$

Major-axis of the oval will be:

$$a = \frac{r_{min} + r_{max}}{2} = 10.1AU$$

According to Kepler Third Law:

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM} = 2.96 \times 10^{-19}$$

We could get, T value is 32.04 years.

We only consider one single path, time will be:

$$t_s = 16.02 \text{ years}$$

And because of energy conservation and major-axis have the relationship with energy:

$$a = \frac{GMm}{2|E|}$$

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{2a} \quad (7.8)$$

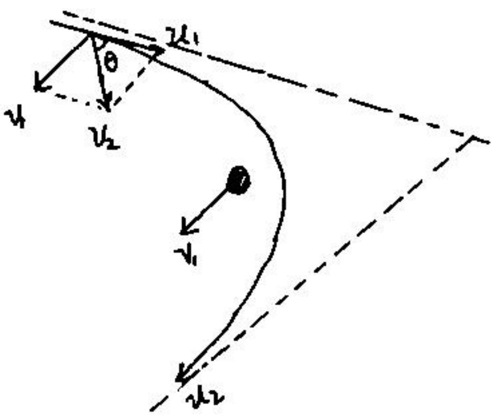
Velocity will be:

$$v = \sqrt{\frac{2GM}{r} - \frac{GM}{a}} \quad (7.9)$$

3.5 TRAVEL WITH JUPITER

3.5.1 FROM JUPITER TO URANUS

Here we consider if the detector uses Jupiter as accelerator:



(Figure 2.9)

This figure shows the orbit of the detector when it passes through Jupiter. The focus point is Jupiter.

In the whole process, the relative speed is constant.

Before the detector gets into Jupiter gravity field it has:

$$P_D = mu_1^2$$

Jupiter has momentum:

$$P_J = Mv_1^2$$

After detector pass it, the momentum change, but momentum is still conservation:

$$-M\Delta v = m(u_2 - u_1)$$

In the same way, energy is also conserved:

$$\frac{1}{2}m{u_1}^2 + \frac{1}{2}M\Delta v^2 = \frac{1}{2}m{u_2}^2 \quad (8.0)$$

Solve the equation so we get:

$$(m+M){u_1}^2 + (m-M){u_2}^2 - 2mu_1u_2 = 0$$

$$m(u_1 - u_2)^2 + M\left(u_1^2 - u_2^2\right) = 0 \quad (8.1)$$

This equation is too rough. We just use it to show how energy and momentum acted on each other but not to calculate.

$$(\overrightarrow{v_1}) \cdot (\overrightarrow{u_1}) = |v_1||u_1|\cos\theta$$

$$\cos\theta = \frac{(\overrightarrow{v_1}) \cdot (\overrightarrow{u_1})}{v_1u_1} = \frac{|v_1||u_1|\cos\theta}{v_1u_1} = \frac{|u_1|\cos\theta}{u_1} \quad (8.2)$$

In this equation, $|u_1|\cos\theta$ term is the detector's velocity component in the total velocity direction, and angular momentum conservation can be set up in this situation:

$$m[|u_1|(\cos\theta)]r_J = mv_Dr_E$$

$$(\cos\theta) = \frac{v_Dr_E}{|u_1|r_J}$$

v_D is the detector speed when it goes around with earth:

$$v_D = \sqrt{\frac{2GM}{r} - \frac{GM}{a}} = 41km/s$$

According to the calculation we can get:

$$\cos\theta = \frac{1}{2}$$

In the same way, when the detector is getting into Jupiter gravitational field, the speed is:

$$u_1 = \sqrt{\frac{2GM}{r_J} - \frac{GM}{a}} = 16km/s$$

And Jupiter speed is:

$$v_1 = 13km/s$$

$$u_2 = \sqrt{u_1^2 + {v_1}^2 - 2u_1v_1\cos\theta} = 14.73km/s$$

When the detector moving out from the field, and take the sun as coordinate original point, the sum of speed is:

$$\overrightarrow{u_{sum}} = \overrightarrow{v_1} + \overrightarrow{u_2}$$

In order to make sure speed is max, $\overrightarrow{v_1}$ and $\overrightarrow{u_2}$ need to parallel with each other:

$$u_{sum} = 14.73 + 13 = 27.73km/s$$

Compared with initial speed, it changes:

$$\Delta u = 11.73km/s$$

Jupiter escape velocity is shown by this:

$$v_e = \sqrt{2}v_J$$

The proof of this is shown in formula (7.2).

$$\frac{v_e}{v_1} = \sqrt{2}$$

According to the calculation:

$$\frac{v_e}{v_1} = \frac{u_{sum}}{v_1} = \frac{27.73}{13} = 2.133 > \sqrt{2}$$

So it can escape, the orbit will be hyperbola.

So orbit formula form is:

$$\frac{L}{r} = 1 + e\cos\theta (e > 0)(8.3)$$

$$r_{min} \approx r_{Jupiter} = \frac{L}{1+e} = 5.2AU$$

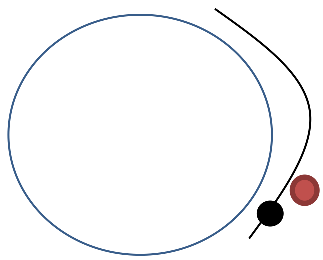
But in formula (8.3) when $r \to \infty$, $e\cos\theta \to -1$.

Based on this, we need to find the value of L.

$$L = \frac{J^2}{m^2GM}$$

This means we need to find the J value.

J value is angular momentum value. This system is for the detector and the sun. Just like the graph shows:



(Figure 3.0)

This graph shows the approximate shape of the detector's orbit after it is accelerated by Jupiter. The black point is the sun, and redpoint is Jupiter, according to figure (1.8). We can say that the eccentricity of Jupiter is 0.049, so the orbit shape (blue curve) is almost a circle. The green line is a hyperbola.

In this situation, just like we mentioned, in the detector-sun system, angular momentum also conserve, but after the elastic collision, one of the planet momentum change to the new angular momentum:

$$J = mu_2 r$$

Here we choose a special point, the hyperbola perihelion. It has maximum speed and shortest radius:

$$J = mu_{sum} r \approx mu_{sum} r_{Jupiter} = 1.76 \times 10^{19}$$

In addition, the new speed we get is the maximum speed, because the direction of the new speed is parallel to the planet's moving direction. In the sun-detector system, at this moment, the detector is in perihelion point. As for minimum radius, we assume $r_{min} \approx r_{Jupiter}$.

Based on now, we could do calculate for L value:

$$L = \frac{J^2}{m^2 GM} = 23.3AU$$

$$\frac{23.3AU}{1+e} = 5.2AU$$

$$e = 3.5$$

Finally, we can get the exact orbit formula:

$$\frac{23.3AU}{r} = 1 + 3.5\cos\theta \ (8.4)$$

This formula is for detector travel from Jupiter to Uranus
when $r = r_{Uran} = 2.9 \times 10^{12} m = 19.1AU$.

$$\frac{23.3AU}{19.1AU} - 1 = 3.5\cos\theta$$

$$\theta_0 = 86.4°$$

Here we take perihelion as polar axis, and this moment also has a maximum speed, so this is a suitable choice. According to the angular momentum formula:

$$t_1 = \int dt = \frac{m}{J}\int_0^{86.4°} (\frac{23.3AU}{1+3.5\cos\theta})^2 d\theta \ (8.5)$$

According to calculate, the approximate value of time:

$$t_1 \approx 3.77 \text{ years}$$

From Earth To Jupiter

This trip is just like figure 2.3 shows. But in this situation, the bigger circle is Jupiter's orbit.

$$\begin{cases} r_{min} = \frac{L}{1+e} = 1AU \\ r_{max} = \frac{L}{1-e} = 5.2AU \end{cases}$$

$$\begin{cases} \frac{r_{min}}{r_{max}} = \frac{1-e}{1+e} = \frac{5}{26} \\ L = 1+e \end{cases}$$

$$\begin{cases} e = 0.677 \\ L = 1.677AU \end{cases}$$

According to Kepler Third Law:

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM} = 2.96 \times 10^{-19}$$

$$a = \frac{r_{min} + r_{max}}{2} = 3.1AU = 4.65 \times 10^{11} m$$

$$\frac{T}{2} \approx 2.73 \text{ years}$$

So total time will be:

$$t_T = 6.5 \text{ years}$$

It can reduce 9.52 years if we use the slingshot effect.

Slingshot Orbit

The spaceship origin speed is 16km/s. We assume that the speed is constant which means before entering Jupiter field. And maximum speed is 37km/s (the sun is the origin point). In the whole process, energy is conserved. When the spaceship is moving to Jupiter field, its speed is 16km/s.

$$\frac{1}{2}mv^2 + \frac{GMm}{r} = \frac{1}{2}mv_{max}^2 - \frac{GMm}{r_{min}}$$

On the left side of the equation, the radius value is big. So it can be considered that $r \to \infty$. Then we can calculate the minimum radius value:

$$r_{min} = \frac{2GM}{v^2 - v_{max}^2} = 1.525 \times 10^{-3}\text{AU}$$

The minimum radius can also be shown like:

$$r_{min} = \frac{L}{1+e} = 1.525 \times 10^{-3}\text{AU}$$

Take Jupiter as the coordinate original point. Use conservation of angular momentum:

$$J = mv_{max}r_{min} = 6.9 \times 10^{15}$$

Then calculate the L value:

$$L = \frac{J^2}{m^2GM} = \frac{4.76 \times 10^{31}}{8.45 \times 10^{22}} = 3.756 \times 10^{-3}\text{AU}$$

Next, we can calculate eccentricity:

$$\frac{3.756 \times 10^{-3}\text{AU}}{1.525 \times 10^{-3}\text{AU}} = 1 + e$$

$$e = 1.463$$

So slingshot orbit function is shown:

$$\frac{3.756 \times 10^{-3}\text{AU}}{r} = 1 + 1.463\cos\theta \ (8.6)$$

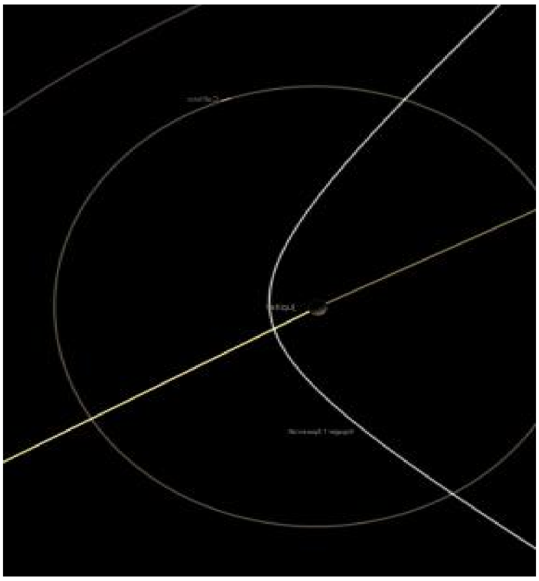
Data Compare And Modeling

(All the simulator pictures which are from figure 2.1 to figure 2.5, figure 3.1 and 3.2 are simulated from Universe Sandbox.)

Table 1
Data of calculating and Voyager1
(in Jupiter orbit)

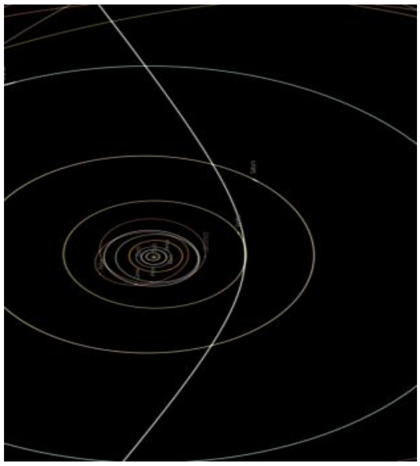
	Voyager1 data	Calculate data
Origin speed(Jupiter)	18 km/s	16km/s
Shot speed(Jupiter)	29km/s	27.73km/s
Time cost travel to Uranus	8 years	6.5 years
Orbit eccentricity (Jupiter)	2.9	3.5

By the comparison of these data in the figure, we can find that these 2 speeds were the most accurate compared with true data. Later on, we will talk about why there will have different errors on different data. Here we list the data of modeling (which is the data on the table).



(Figure 3.1)

Jupiter to Uranus curve and data:



(Figure 3.2)

In figure (3.2), the focus of the hyperbola is the sun.

And we could use the Voyager 1 data to calculate the accurate orbit formula.

$$L = r_{min}(1 + e) = 5.16AU \times 3.9 = 20.12AU$$

So the Voyager orbit formula will be:

$$r = \frac{20.12AU}{1 + 2.9\cos\theta} \quad (8.7)$$

To be specific, the value of eccentricity is variable, here we use the most stable value which is 2.9. The figure shows a smaller value. 3 orbits function are shown below:

From Earth to Jupiter:

$$\frac{1.9\text{AU}}{r} = 1 + 0.9\cos\theta$$

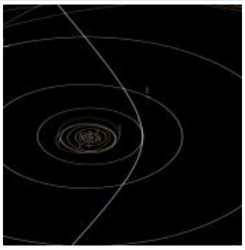
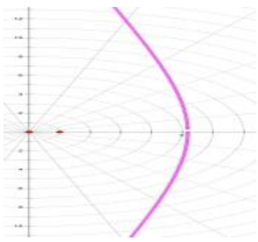
Slingshot with Jupiter:

$$\frac{3.756 \times 10^{-3}\text{AU}}{r} = 1 + 1.463\cos\theta$$

From Jupiter to Uranus:

$$\frac{23.3\text{AU}}{r} = 1 + 3.5\cos\theta$$

Table 2 Orbit shapes in Jupiter to Uranus

	Voyager1 orbit	Calculate orbit
Shape		
Function	$r = \frac{20.12\text{AU}}{1 + 2.9\cos\theta}$	$r = \frac{23.3\text{AU}}{1 + 3.5\cos\theta}$

Fuel Consumption

Here we will have a brief and simple discussion about how much fuel we can save by using the slingshot effect.

When a spaceship gets into the planet's gravitational field, gravity will do the positive work for the spaceship instead of the engine. It will get a higher speed, but with longer the spaceship travel, the speed of this will get slower, when it speeds to get the original speed (the speed when spaceship gets into the gravitational field). Then the engine starts work. So during the process, the fuel consumption will get lower. What we will do is to find how much energy can be saved, and only depends on gravity.

Saving energy is:

$$E_{save} = \frac{1}{2}m\left(v_{max}^2 - v_{origin}^2\right) = 4.53 \times 10^{11}\text{J}$$

In addition, when a spaceship gets out of the gravitational field, the focus of the curve is the sun, and because of that, kinetic energy will almost have no loss.

Analysis

1. The reason why the speed will have errors is that we assume the planet orbit is a circle (e=0), and in the travel, we didn't consider the loss of energy. And most of the calculation is math calculation, we did not consider too much in a true universe situation.

2. Eccentricity has the same theory as speed. The value of it will get smaller and smaller with the longer distance. In our calculation, we assume that eccentricity is a constant, we did not consider the variable time in, which means the value we calculated will be bigger than the true value and that is the most main reason.

3. For the whole time, the true value is bigger. As we mentioned, eccentricity is higher actually and the focal chord is also smaller. Time can be calculated by integration for radius function, so in the slingshot process, the time will be longer actually, and also spacecraft needs correct orbit that also takes a long time.

4. Before and after the process of the slingshot, the orbit shape will change. During the slingshot process, the orbit shape is shown in

figure (3.1). After the slingshot, the orbit shape is shown in figure (3.2). So when we do the calculation, we need to notice this. The reason for this is because when the spaceship has done the slingshot, it will get out of the Jupiter gravitational field, the effect of Jupiter will be weaker. So that the sun will be the new focus.

5. Slingshot effect can change spaceship direction. It can also make it speed up or speed down. The key to this effect is energy conservation and angular momentum conservation.

6. The main problem is to find the correct coordinate system.

Conclusion

In this thesis, we made a discussion for the Two-body problem and solve the Slingshot problem.

For the first part, we proved Kepler's three laws and show some Two-body models in the universe (binary stars and binary black holes.) and the key of the two-body problem is conservation of angular momentum. Parameter equations and Polar coordinate systems are two important ways of solving problems.

For the second part, we use gravitational potential energy correction and Einstein Special and General relativity to calculate the perihelion precession angle value.

For potential correction:

First, we add a small addiction potential energy then expanded it in order to get a general formula include this addiction potential energy- $\delta V(r)$. Then we assume all extra planets' potential energy was averagely distributed on a plate. Then use energy integration to calculate the approximate value of this addiction energy. Put the value into the formula then get the final result.

For special relativity correction:

First, we expanded the relativity momentum formula to a third term, and make approximate calculations for energy, then use the conservation of energy to solve a differentiation function, compared it with Part 1 classical Function, and get the correction of special relativity. Then we find Einstein theory correction only takes almost 8% parts of the whole value. Tylor Expansion plays an important role in it.

In the third part, we introduced the Slingshot effect then we calculate the Voyager1 orbits (Before and after using Jupiter to do Slingshot, and Slingshot orbit) and we also calculate speed, eccentricity and time. Then compared them with true values, the error range is about 5% to 11.45%. We also simply proved that using these effects can save lots of energy.

Shortage Of Research

In part one: We did not calculate the final integration.

For part two: Gravitational potential Correction is based on all the potential energy of planets that were distributed averagely on a range of space, but this assumption didn't work in Venus orbit so that is a shortage. For the process of relativity correction, we only calculated the special relativity (because personal physics and math skills did not good). So as for general relativity correction, we just use the formula to calculate.

For part three: We did not consider the eccentricity of earth and Uranus. We just take them as circular orbit. This will lead to errors. The explanation for why energy is negative is not perfect. In fact, it has some mistakes.

For part three: We did not consider the change of eccentricity and speed of the detectors. In the variable coordinate systems, the expression isn't very clear. The way of getting orbit just one single way. And when we calculate time, slingshot time did not consider. Fuel consumption calculations are too easy. We just use kinetic energy difference to do it, and the assumption is too ideality.

Conflict of Interests: the author has claimed that no conflict of interests exists.

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